The equations (82) and (83) assign the effect of any homogeneous mass symmetrical with respect to an axis, subject to the restriction that the maximum thickness of the mass may be neglected in comparison with the radius of the earth.

The integral in (82) depends on, and will in general be, no less complex than I, which is defined by equations (22) to (25). In the application of (83) it is to be observed that by (46)

$$\frac{dF_i(\beta)}{d\beta} = f_i(\cos\beta)\,.\,\sin\beta.$$

SOLUTIONS OF EXERCISES.

94

O is the centre of the circumscribed circle of ABC, and D, E, F the middle points of its sides. Show that

$$OD^2 + OE^2 + OF^2 = 2R'(2R' - r'),$$

where R', r' are the radii of the circumscribed and inscribed circles of the triangle of the feet of the altitudes. [*R. D. Bohannan.*]

SOLUTION.

Let H be the orthocentre, and A', B', C' the feet of the perpendiculars. Since $OD = R\cos A$, etc.,

$$OD^2 + OE^2 + OF^2 = R^2(\cos^2 A + \cos^2 B + \cos^2 C)$$
$$= R^2(1 - 2\cos A\cos B\cos C).$$

But $\quad R = 2R', \quad$ and $\quad r' = A'H\cos A$

$$= BH\cos A\cos C$$
$$= 2R\cos A\cos B\cos C;$$

$$\therefore \quad OD^2 + OE^2 + OF^2 = R^2\left(1 - \frac{r'}{R}\right) = 2R'(2R' - r').$$

[*Marcus Baker.*]

97

IN the triangle ABC two lines drawn from C trisect the side AB. Given c, C, and the angle φ between the trisecants; to solve the triangle.

[*Marcus Baker.*]

SOLUTION.

Let M, N be the points of trisection. Then the anharmonic ratios of the row $AMNB$ and the pencil C–$AMNB$ being equal, we have

$$\frac{\sin ACM \cdot \sin BCN}{\sin ACB \cdot \sin MCN} = \frac{AM \cdot BN}{AB \cdot MN},$$

or, with the notations given,

$$\sin x \sin y = \tfrac{1}{3} \sin C \sin \varphi,$$

where $x = ACM, \quad y = BCN.$

We have also $x + y = C - \varphi.$

But since $\cos(x - y) = \cos(x + y) + 2 \sin x \sin y,$

these relations give

$$\cos(x - y) = \tfrac{4}{3} \cos(C - \varphi) - \tfrac{1}{3} \cos(C + \varphi);$$

whence $x - y$ is to be found, and thence x and y. A and B are then found from the relations

$$\tan \tfrac{1}{2}(A - N) = \frac{\tan \frac{1}{2}(x - \varphi)}{\tan^2 \frac{1}{2}(x + \varphi)}, \quad A + N = \pi - x - \varphi;$$

$$\tan \tfrac{1}{2}(B - M) = \frac{\tan \frac{1}{2}(y - \varphi)}{\tan^2 \frac{1}{2}(y + \varphi)}, \quad B + M = \pi - y - \varphi.$$

[*J. E. Hendricks.*]

98

THE eccentric anomalies of three points on an ellipse are p_1, p_2, p_3. Show that the area of their triangle is

$$\Delta = 2ab \sin \frac{p_2 - p_3}{2} \sin \frac{p_3 - p_1}{2} \sin \frac{p_1 - p_2}{2};$$

the centre of its circumscribing circle

$$x = + \frac{c^2}{a} \cos \frac{p_2 + p_3}{2} \cos \frac{p_3 + p_1}{2} \cos \frac{p_1 + p_2}{2},$$

$$y = - \frac{c^2}{b} \sin \frac{p_2 + p_3}{2} \sin \frac{p_3 + p_1}{2} \sin \frac{p_1 + p_2}{2};$$

and hence show that the centre of curvature of the ellipse at (x, y) is

$$X = + \frac{c^2 x^3}{a^4}, \quad Y = - \frac{c^2 y^3}{b^4}.$$

[*W. M. Thornton.*]

SOLUTION I.

The co-ordinates of the three points are

$$x_1 = a\cos p_1,\quad y_1 = b\sin p_1;$$
$$x_2 = a\cos p_2,\quad y_2 = b\sin p_2;$$
$$x_3 = a\cos p_3,\quad y_3 = b\sin p_3.$$

Then* $\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_2 & y_3 - y_2 \end{vmatrix}$

$$= \tfrac{1}{2}ab\begin{vmatrix} \cos p_2 - \cos p_1 & \sin p_2 - \sin p_1 \\ \cos p_3 - \cos p_2 & \sin p_3 - \sin p_2 \end{vmatrix}$$

$$= -2ab\begin{vmatrix} \sin\tfrac{1}{2}(p_1+p_2)\sin\tfrac{1}{2}(p_1-p_2) & \cos\tfrac{1}{2}(p_1+p_2)\sin\tfrac{1}{2}(p_1-p_2) \\ \sin\tfrac{1}{2}(p_2+p_3)\sin\tfrac{1}{2}(p_2-p_3) & \cos\tfrac{1}{2}(p_2+p_3)\sin\tfrac{1}{2}(p_2-p_3) \end{vmatrix}$$

$$= 2ab\sin\tfrac{1}{2}(p_1-p_2)\sin\tfrac{1}{2}(p_2-p_3)\sin\tfrac{1}{2}(p_3-p_1).$$

The co-ordinates of the centre† are

$$x = \quad \tfrac{1}{4}\begin{vmatrix} 1 & y_1 & r_1^2 \\ 1 & y_2 & r_2^2 \\ 1 & y_3 & r_3^2 \end{vmatrix} \div \Delta = \tfrac{1}{4}\alpha / \Delta,$$

$$y = -\tfrac{1}{4}\begin{vmatrix} 1 & x_1 & r_1^2 \\ 1 & x_2 & r_2^2 \\ 1 & x_3 & r_3^2 \end{vmatrix} \div \Delta = \tfrac{1}{4}\beta / \Delta.$$

Substituting the values of the co-ordinates of the three points,

$$\alpha = b\begin{vmatrix} 1 & \sin p_1 & a^2 - (a^2 - b^2)\sin^2 p_1 \\ 1 & \sin p_2 & a^2 - (a^2 - b^2)\sin^2 p_2 \\ 1 & \sin p_3 & a^2 - (a^2 - b^2)\sin^2 p_3 \end{vmatrix}$$

$$= -b(a^2 - b^2)\begin{vmatrix} 1 & \sin p_1 & \sin^2 p_1 \\ 1 & \sin p_2 & \sin^2 p_2 \\ 1 & \sin p_3 & \sin^2 p_3 \end{vmatrix}$$

$$= b(a^2 - b^2)(\sin p_1 - \sin p_2)(\sin p_2 - \sin p_3)(\sin p_3 - \sin p_1)$$

$$= 8b(a^2 - b^2)\cos\tfrac{1}{2}(p_1+p_2)\cos\tfrac{1}{2}(p_2+p_3)\cos\tfrac{1}{2}(p_3+p_1)\sin\tfrac{1}{2}(p_1-p_2)$$
$$\sin\tfrac{1}{2}(p_2-p_3)\sin\tfrac{1}{2}(p_3-p_1).$$

* Salmon's *Conic Sections*, § 36.

† Salmon's *Conic Sections*, §§ 80, *94.

Similarly

$$\beta = a \begin{vmatrix} 1 & \cos p_1 & b^2 + (a^2 - b^2)\cos^2 p_1 \\ 1 & \cos p_2 & b^2 + (a^2 - b^2)\cos^2 p_2 \\ 1 & \cos p_3 & b^2 + (a^2 - b^2)\cos^2 p_3 \end{vmatrix}$$

$$= -8a(a^2 - b^2)\sin\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_2 + p_3)\sin\tfrac{1}{2}(p_3 + p_1)\sin\tfrac{1}{2}(p_1 - p_2)$$
$$\sin\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_3 - p_1).$$

Hence, if $c^2 = a^2 - b^2$,

$$x = +\frac{c^2}{a}\cos\tfrac{1}{2}(p_1 + p_2)\cos\tfrac{1}{2}(p_2 + p_3)\cos\tfrac{1}{2}(p_3 + p_1),$$

$$y = -\frac{c^2}{b}\sin\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_2 + p_3)\sin\tfrac{1}{2}(p_3 + p_1).$$

If the three points coincide, these become

$$X = +\frac{c^2}{a}\cos^3 p = +\frac{c^2x^3}{a^4},$$

$$Y = -\frac{c^2}{b}\sin^3 p = -\frac{c^2y^3}{b^4}.$$

[*Ormond Stone.*]

SOLUTION II.

$$2\Delta = (y_1 - y_2)(x_3 - x_2) - (x_1 - x_2)(y_3 - y_2)$$
$$= ab\,[(\sin p_1 - \sin p_2)(\cos p_3 - \cos p_2) - (\sin p_3 - \sin p_2)(\cos p_1 - \cos p_2)]$$
$$= 4ab\,[-\cos\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_1 - p_2)\sin\tfrac{1}{2}(p_3 + p_2)\sin\tfrac{1}{2}(p_3 - p_2)$$
$$+ \cos\tfrac{1}{2}(p_3 + p_2)\sin\tfrac{1}{2}(p_3 - p_2)\sin\tfrac{1}{2}(p_1 + p_2)\sin\tfrac{1}{2}(p_1 - p_2)]$$
$$= 4ab\sin\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_1 - p_2)\,[\sin\tfrac{1}{2}(p_2 + p_3)\cos\tfrac{1}{2}(p_1 + p_2)$$
$$- \cos\tfrac{1}{2}(p_2 + p_3)\sin\tfrac{1}{2}(p_1 + p_2)]$$
$$= 4ab\sin\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_1 - p_2)\sin\tfrac{1}{2}(p_3 - p_1).$$

The equations of the perpendiculars to the chords at their middle points are

$$y - \tfrac{1}{2}(y_1 + y_2) = -\frac{x_2 - x_1}{y_2 - y_1}[x - \tfrac{1}{2}(x_1 + x_2)]$$

$$y - \tfrac{1}{2}(y_2 + y_3) = -\frac{x_3 - x_2}{y_3 - y_2}[x - \tfrac{1}{2}(x_2 + x_3)].$$

Eliminating y, we have

$$\tfrac{1}{2}(y_3 - y_1) = x\left(\frac{x_3 - x_2}{y_3 - y_2} - \frac{x_2 - x_1}{y_2 - y_1}\right) + \frac{(x_2 - x_1)(x_2 + x_1)}{2(y_2 - y_1)} - \frac{(x_3 - x_2)(x_3 + x_2)}{2(y_3 - y_2)};$$

whence, after substituting the values of x_1, y_1, etc., and making suitable reductions, we find

$$x = +\frac{c^2}{a}\cos\tfrac{1}{2}(p_2+p_3)\cos\tfrac{1}{2}(p_3+p_1)\cos\tfrac{1}{2}(p_1+p_2).$$

In like manner, by eliminating x;

$$y = -\frac{c^2}{b}\sin\tfrac{1}{2}(p_2+p_3)\sin\tfrac{1}{2}(p_3+p_1)\sin\tfrac{1}{2}(p_1+p_2).$$

Suppose (x_1, y_1) and (x_3, y_3) to approach (x_2, y_2), then the limiting position of the centre of the circumscribing will be the centre of curvature at (x_2, y_2). But at the limit,

$$p_1 = p_2 = p_3 = p, \quad \cos p = \frac{x}{a}, \quad \text{and} \quad \sin p = \frac{y}{b};$$

$$\therefore \quad X = \frac{c^2x^3}{a^4}, \quad Y = -\frac{c^2y^3}{b^4}.$$ [*Charles Puryear.*]

99

From a full cask of wine a quantity is taken out at random and the cask filled with water, and then a quantity of the mixture is taken out at random and the cask again filled with water. What is the probability that the cask now contains more wine than water? [*Artemas Martin.*]

SOLUTION.

Let x = quantity of wine taken out at the first drawing, y = quantity of the mixture taken out at the second drawing. Then, a being the capacity of the cask, $a\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)$ = quantity of wine remaining in the cask after the second drawing. Putting

$$a\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right) = \tfrac{1}{2}a, \tag{1}$$

we get
$$x = a\left\{1 - \frac{1}{2\left(1-\frac{y}{a}\right)}\right\} = x_1.$$

Now the greater x is, the less the amount of wine left; hence x may have any value from 0 to x_1.

$$\therefore \quad p' = \frac{x_1}{a} = 1 - \frac{1}{2\left(1-\frac{y}{a}\right)}$$

is the probability on the supposition that y is known.

The greatest value y can have is found by putting $x = 0$ in (1), which gives $y = \frac{1}{2}a$. Hence the required probability is

$$p = \frac{\int_0^{\frac{1}{2}a} p'dy}{\int_0^{a} dy} = \frac{1}{a}\int_0^{\frac{1}{2}a}\left\{1 - \frac{1}{2\left(1 - \frac{y}{a}\right)}\right\}dy$$

$$= \frac{1}{a}\left[y + \tfrac{1}{2}a \log\left(1 - \frac{y}{a}\right)\right]_0^{\frac{1}{2}a}$$

$$= 1 - \tfrac{1}{2}(1 + \log 2).$$

[*Artemas Martin.*]

101

EXPRESS in terms of $\sin^{-1}x$ and $\sin^{-1}y$

$$\tan^{-1}\frac{x+y}{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}}.$$

SOLUTION.

If $\sin^{-1}x = a$ and $\sin^{-1}y = b$, we have

$$\tan^{-1}\frac{x+y}{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}} = \tan^{-1}\frac{\sin a + \sin b}{\cos a + \cos b}$$

$$= \tan^{-1}\tan\tfrac{1}{2}(a+b) = \tfrac{1}{2}(a+b) = \tfrac{1}{2}(\sin^{-1}x + \sin^{-1}y).$$

[*Cooper D. Schmitt.*]

EXERCISES.

111

A circle is inscribed in an isosceles triangle; another in the space between the first circle and the vertex, and so on *ad infinitum*. What is the vertical angle when the sum of the areas of the circles is $1/n$ of the area of the triangle?

[*L. G. Weld.*]

112

To inscribe in the given triangle ABC a triangle PQR whose sides QR, RP, PQ are perpendicular respectively to BC, CA, AB. [*Alfred C. Lane.*]